ASTRON 449, Winter 2019 – Problem Set 5

Due Thu March 14, in class.

REGULAR PROBLEMS:

1. Epicyclic frequency for different rotation curves. Evaluate the angular frequency Ω and epicyclic frequency κ for the following rotation curves. In each case, express κ in terms of Ω:
a) $v_{\rm c}(r)$=const. (flat)
b) $v_{\rm c}(r) = \sqrt{GM/r}$ (Keplerian)
c) $v_{\rm c}(r) = v_{\rm c}(R_0)(R/R_0)$ (linear)

2. Self-gravitating $Q = 1$ disk model. A useful (albeit not completely self-consistent) model for galaxies with flat rotation curve assumes that a gas disk is in radial centrifugal balance in an isothermal potential with velocity dispersion σ. Define the total mass and the gas mass within a radius r as $M_{\rm tot}(r)$ and $M_{\rm g}(r)$, respectively. Surface densities are defined as $\Sigma_x(r) \equiv M_x(r)/\pi r^2$ and we define the gas mass fraction $f_{\rm g} \equiv \Sigma_{\rm g}/\Sigma_{\rm tot}$. We assume that $f_{\rm g}$ is a constant.

a) Show that the enclosed mass $M_{\rm tot}(r) = 2\sigma^2 r/G$, that the circular velocity $v_{\rm c} = \sqrt{2}\sigma$, and that the angular frequency $\Omega = \sqrt{2}\sigma/r$.

b) For a thin gas disk in a spherical gravitational potential, show that the equation of hydrostatic equilibrium is (to first order)

$$\frac{\partial P}{\partial z} = -\rho\Omega^2 z, \tag{1}$$

where P is the gas pressure, ρ is the gas density, and z is the coordinate normal to the disk plane.

c) Argue that this can be approximated (to order unity) by

$$P \approx \bar{\rho} h^2 \Omega^2, \tag{2}$$

where h is the disk scale height.

d) If turbulence dominates the effective gas pressure, $P \approx P_{\rm T} \approx \bar{\rho} c_{\rm T}^2$, where $c_{\rm T}$ is the turbulent velocity. Thus show that the scale height of the disk is set by the ratio of the turbulent velocity to the angular frequency in the disk,

$$h \approx \frac{c_{\rm T}}{\Omega}. \tag{3}$$

e) There is strong observational evidence and support from numerical experiments that star-forming galactic disks self-regulate to a Toomre Q parameter near unity,

$$Q = \frac{\kappa c_{\rm T}}{\pi G \Sigma_{\rm g}} \approx 1, \tag{4}$$

where κ is the epicyclic frequency. This can be understood intuitively from the fact that galactic disks tend to cool (dissipate their turbulent energy) until $Q \approx 1$ at which point they become gravitationally unstable and form stars. Stellar feedback then drives turbulence in the disk, ensuring that Q does not drop significantly below unity.

Evaluate the epicyclic frequency for the isothermal potential and express Q in terms of σ and r.

f) Neglecting the contribution of elements heavier than hydrogen, we may define the mean hydrogen number density in the disk as $\bar{n}_{\rm H} \equiv \bar{\rho}/m_{\rm p}$. Show that

$$\bar{n}_{\rm H} = \frac{\sqrt{2}\sigma^2}{\pi G Q r^2 m_{\rm p}}. \tag{5}$$

Ultra-luminous infrared galaxies (ULIRGs) have gas-rich nuclear disks of size $r \approx 100$ pc and velocity dispersion $\sigma \approx 200$ km/s. Evaluate the mean hydrogen number density using these parameters and compare with the mean interstellar medium density $\bar{n}_{\rm H} \approx 1$ cm^{-3} in the solar neighborhood.

g) Using the definitions of Q and $f_{\rm g}$, prove the following three useful relationships between disk scale height and radius, and disk turbulent velocity and potential velocity dispersion:

$$\frac{h}{r} = \frac{Q}{2^{3/2}} f_{\rm g}; \qquad \frac{c_{\rm T}}{\sigma} = \frac{Q}{2} f_{\rm g}; \qquad \frac{h}{r} = \frac{1}{\sqrt{2}} \frac{c_{\rm T}}{\sigma}. \tag{6}$$

To get the correct dimensionless pre-factors, you should assume that the disk scale height is related to the gas surface density via $\Sigma_{\rm g} = 2h\bar{\rho}$.

For a disk with $Q \approx 1$, these equations show that the disk thickness and turbulent velocity dispersion are set by the gas mass fractions $f_{\rm g}$. Massive local spiral galaxies have typical $f_{\rm g} \approx 10\%$ while massive star-forming galaxies at high redshift ($z = 2$) typically have $f_{\rm g} \approx 50\%$. As equation (6) predicts, $z = 2$ star-forming galaxies are observed to have disk thicknesses and turbulent velocity dispersions a factor ≈ 5 higher than local galaxies.

3. Winding of spiral arms and application to swing amplification. Consider the spiral arm geometry in Figure 1, where the pitch angle α is defined. Assume that (aside from deviations due to the spiral arm) stars are in circular orbits around the galaxy.

a) Show that the pitch angle satisfies

$$\cot \alpha = \left| R \frac{\partial \phi}{\partial R} \right|, \tag{7}$$

where the partial derivative is evaluated along the spiral arm.

b) Suppose that at $t = 0$, we paint a narrow stripe or arm radially outward across the disk. This is what we called a 'material' arm in class, because it follows a fixed set of stars. The initial equation for the stripe is $\phi = \phi_0$, where ϕ is the azimuthal angle. The disk rotates with angular

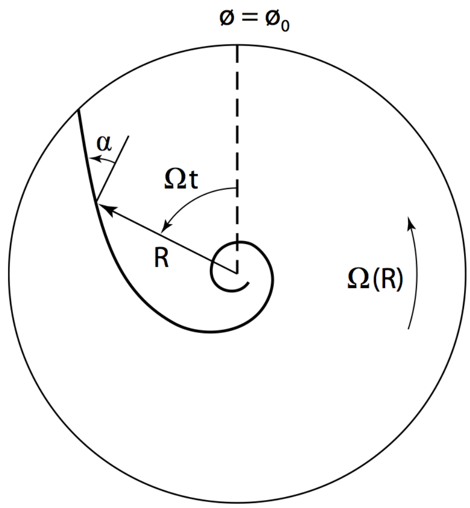

Fig. 1.— Geometry for problem 2. The pitch angle α at any radius R is the angle between the tangent to the arm and the circle $R = const$. By definition, $0 < \alpha < 90^\circ$ for trailing arms (like the one shown). Leading arms are defined such that $90 < \alpha < 180^\circ$.

frequency $\Omega(R)$. When the disk is in differential rotation (Ω varying with R), the arm winds as the disk rotates. Show that the pitch angle of the material after time t is

$$\cot\alpha = -Rt\frac{d\Omega}{dR} = 2At, \tag{8}$$

where $A \equiv -(1/2)Rd\Omega/dR$ is one of Oort's constants and quantifies shear in the disk.

c) Evaluate α numerically for a representative disk galaxy with a flat $v_c = 200$ km s^{-1} rotation curve, $R = 5$ kpc, and $t = 10$ Gyr. Compare your answer with the typical pitch angle $\alpha = 15^\circ$ observed in spiral galaxies. Can observed spirals trace material arms?

d) Show that the rate of change of the pitch angle is

$$\frac{d\alpha}{dt} = -\frac{2A}{1 + 4A^2t^2}. \tag{9}$$

e) When the arm is tightly wound (corresponding to $t \to \pm\infty$), its rotation rate $d\alpha/dt$ is slow, but as it swings from leading to trailing at $t = 0$ it reaches a maximum rotation rate of $2A$. For a galaxy with flat rotation curve, show that $2A \sim \Omega \sim \kappa$.

Thus there is a temporary near-match between epicyclic motion and the rotating spiral feature (both of which are in the same sense), which enhances the effect of the gravitational force from the spiral on the stellar orbit – and the contribution of the star's own gravity to the spiral perturbation. This is what we call swing amplification.

f) Do initially trailing perturbations get swing amplified like initially leading ones? Explain your answer.

4. Orbital decay due to dynamical friction. Approximate the density distribution of a galaxy as a singular isothermal sphere,

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}, \tag{10}$$

with Maxwellian velocity distribution.

a) Evaluate the dynamical friction force experienced by an object of mass M in a circular orbit at radius r. Show that the answer can be expressed in terms of M, r, and $\ln\Lambda$ only, with no explicit dependence on σ. Simplify your answer as much as possible.

b) Use the fact the subject mass M spirals in toward the center of the galaxy following a series of nearly circular orbits to evaluate the time $t_{\rm fric}$ for M to reach $r = 0$. Your answer should involve only $\ln\Lambda$ (which you can assume is constant), the initial radius $r_{\rm i}$, σ, and M.

c) Suppose that a black hole of mass $M = 10^8$ $\rm M_\odot$ is initially at radius $r_{\rm i} = 5$ kpc from the center of a galaxy of velocity dispersion $\sigma = 200$ km s^{-1}. Evaluate $t_{\rm fric}$ numerically for this black hole. By comparing to the typical age 10 Gyr of galaxies, what can you say about where massive black holes should be found in galaxies today?

d) Consider now globular clusters with a typical mass $M = 2 \times 10^5$ $\rm M_\odot$, also in orbit in a galaxy with $\sigma = 200$ km s^{-1}. Evaluate the radius within which $t_{\rm fric} < 10$ Gyr for such globular clusters. According to your result, is it surprising that globular clusters are numerous in the halos of galaxies?

e) Consider a cosmological $N-$body (+ hydrodynamics) simulation following the formation of a Milky Way-like galaxy. The Milky Way's central supermassive black hole has a mass $M_{\rm BH} = 4 \times 10^6$ $\rm M_\odot$ and the simulation has resolution elements of mass $m_{\rm b} = 10^6$ $\rm M_\odot$ (this is representative of many state-of-the-art large-volume simulations). Explain why this simulation cannot self-consistently capture the dynamical friction acting on the supermassive black hole and hence cannot reliably predict the trajectory of the black hole (e.g., as progenitor galaxies merge and their black holes are displaced from galaxy center).

COMPUTATIONAL PROBLEMS:

Reminder concerning units: Treat Newton's constant G as a variable whose value can be modified in the code. By default, we work in dimensionless units and set $G = 1$.

The Toomre analysis that we covered in class is for the *local* stability of infinitely thin rotating disks to *axisymmetric* perturbations. In order to investigate the *global* stability of disks and to follow the *non-linear* development of gravitational instabilities seeded by arbitrary perturbations, it is in general necessary to use numerical simulations.

In this problem set, you will use the 2D PM code that you developed in PS4 to evolve a few illustrative disk simulations. You will also learn how to generate initial conditions for such simulations.

C1. Generating ICs: 2D Mestel disk+halo model. We construct a 2D model consisting of a disk of "live" disk particles (the $N-$body particles) along with a "fixed" gravitational potential which mimics the effects of a dark matter halo or large bulge. To do this, we use properties of the Mestel disk model, which is discussed in BT2, §2.6.1a.

For the live stellar disk, the initial mass surface density is parameterized by

$$\Sigma_{\rm d}(R) = \begin{cases} \frac{f_{\rm d} v_{\rm c}^2}{2\pi G R} & \text{if } R \leq R_{\rm max} \\ 0 & \text{if } R > R_{\rm max}, \end{cases} \tag{11}$$

where $f_{\rm d}$ is the disk fraction parameter. As $R_{\rm max} \to \infty$, this surface density is that of a Mestel disk with constant circular velocity $\sqrt{f_{\rm d}} v_{\rm c}$. A property of the Mestel disk is that the centripetal acceleration at radius R is a function only of the enclosed mass $M(R)$:[1] $a_{\rm centripetal,d}(R) = f_{\rm d} v_{\rm c}^2/R$.

We want the centripetal acceleration due to the total mass distribution (including the fixed halo) to produce a constant circular velocity $v_{\rm c}$ within the disk, so we set the fixed halo centripetal acceleration to $a_{\rm centripetal,h}(R) = (1 - f_{\rm d})v_{\rm c}^2/R$. Then, the total centripetal acceleration within $R_{\rm max}$ is $a_{\rm centripetal,tot} = a_{\rm centripetal,d} + a_{\rm centripetal,h} \approx v_{\rm c}^2/R$, so disk particles initialized on circular orbits with velocity $v_{\rm c}$ will initially satisfy force balance.

a) For a pure stellar disk, the Toomre Q parameter is

$$Q = \frac{\sigma_R \kappa}{3.36 G \Sigma_{\rm d}}, \tag{12}$$

where σ_R is the one-dimensional velocity dispersion in the radial direction. The 3.36 pre-factor in the denominator (rather than π) is obtained when performing the stability analysis for a stellar disk rather than a fluid disk. In this expression, the epicyclic frequency κ should be evaluated for the total potential (including the halo component), since the total potential sets the disk rotation.

Derive an expression for σ_R as a function of Q and $v_{\rm c}$ valid within $R_{\rm max}$, for the above Mestel disk+halo model.

b) Write a Python program to generate initial conditions for the live stellar disk as a function of the following parameters: $R_{\rm max}$, $v_{\rm c}$, Q (used to evaluate $\sigma_{\rm R}$), $f_{\rm d}$, and N. Your program should produce a list of particle positions and velocities randomly sampling the phase-phase distribution function of the initial disk and write them to an ASCII file in the format that can be used as input to your `pm2d.py` program:

[1] This is always true for spherically-symmetric mass distributions in 3D, but it is not in general true for axisymmetric disks.

```
1 m1 x1 y1 vx1 vy1
2 m2 x2 y2 vx2 vy2
...
N mN xN yN vxN vyN.
```

Assume that all N particles have the same mass m.

The initial spatial distribution of disk particles should follow equation (11) for the mass surface density. To see how particles sample the disk in radius, evaluate $dM_{\rm d}/dR$ and use the result to randomly assign R coordinates. Since the disk is axisymmetric, the polar angle coordinate ϕ (such that $x = R\cos\phi$ and $y = R\sin\phi$) should uniformly sample angles ranging from 0 to 2π.

Within the disk, we assume that the initial velocity distribution consists of counter-clockwise circular orbits with velocity $v_{\rm c}$, on top of which we add Gaussian velocity perturbations, so that $Q > 0$. Assuming that the velocity perturbations are isotropic, we have $\sigma_{v_x} = \sigma_{v_y} = \sigma_R$ and orthogonal velocity perturbation components are independent.

Show that these assumptions imply that the initial velocity distribution function at any point in the disk can be expressed as:

$$f(v_x,\ v_y)dv_x dv_y = f(v_x)f(v_y)dv_x dv_y, \tag{13}$$

where

$$f(v_x)dv_x = \frac{1}{\sqrt{2\pi}\sigma_{\rm R}} \exp\left[-\frac{(v_x + v_{\rm c}\sin\phi)^2}{2\sigma_{\rm R}^2}\right] dv_x \tag{14}$$

and

$$f(v_y)dv_y = \frac{1}{\sqrt{2\pi}\sigma_{\rm R}} \exp\left[-\frac{(v_y - v_{\rm c}\cos\phi)^2}{2\sigma_{\rm R}^2}\right] dv_y. \tag{15}$$

To randomly assign positions and velocities to particles following the desired distributions, you can use functions in Python's `random` module. The functions `random.random()` and `random.gauss()`, in particular, may be useful.

c) To implement the fixed halo potential, modify your `pm2d.py` code so that the total acceleration is a sum of the acceleration due to the live particles and a fixed central force field, i.e. $\mathbf{a}_{\rm tot} = \mathbf{a}_{\rm pm} + \mathbf{a}_{\rm fixed}$, where $\mathbf{a}_{\rm pm}$ is the acceleration evaluated using the PM method while $\mathbf{a}_{\rm fixed}$ is a term evaluated analytically. Note that the total energy of the system should then include a component due to the fixed potential.

You can hardcode parameters of the $\mathbf{a}_{\rm fixed}$ function, i.e. modify the parameters directly in the Python source file for different simulations rather than add command line options.

C2. Disk simulations. You will now produce initial conditions for and evolve four different simulations with varying Toomre Q parameter and disk fraction to study the effects of these parameters on the development of a stellar disk.

In all simulations, keep the following parameters the same:

- $N = 10^5$ particles, a PM mesh with 1024 grid nodes along each direction in the physical domain, physical width of the PM mesh (`gridsize`) = 10, and a Plummer softening length $\epsilon = 0.05$. This spatial resolution is necessary in order to capture the fine-scale structure that develops in some of the simulations.
- $R_{\rm max}$=3 and $v_{\rm c} = 2\pi$. This implies that the orbital period is unity at $R = 1$ (longer at larger R).
- Leapfrog integrator with timestep $dt = 0.01$.

Evolve each simulation for a time $t > 6$ and write to disk enough particle snapshots to plot the data at $t = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6$ to see how the disk structure evolves with time. Without much optimization, my code took $\sim$ 40 mins to run each simulation with these parameters (you can evolve a few simulations at the same time on multicore computers). Note that simulations with this large N would have been impossible to evolve with your direct summation code!

For each simulation, produce a plot following the example in Figure 2 to show how the disk structure develops from $t = 0$ to $t = 6$, as well as energy and angular momentum conservation diagnostics to help validate your results. Tip: Instead of a scatter plot with 10^5 points in each panel, use the matplotlib function `hexbin` to plot your disks as 2D histograms. This will make your plots much smaller and can also help show structures better.

Following are the parameters to vary for your four simulations:

- Default model ($Q > 1$, large halo fraction): $Q = 1.25$ and $f_{\rm d} = 0.15$.
- Low-Q model: $Q = 0.75$ and $f_{\rm d} = 0.15$.
- No-halo model: $Q = 1.25$ and $f_{\rm d} = 1$.
- Massive perturber model. Default model but with one additional massive particle inserted in the initial conditions, modeling the encounter with a massive perturber (e.g., another galaxy). For the massive perturber, use the following parameters: $m = 10$ (how does this compare to the mass of the stellar disk?), $x_0 = 4.9$, $y_0 = 4.5$, $v_{x,0} = -5$, $v_{y,0} = 0$.

By comparing the results of your four simulations, comment on the following (one-sentence answers): 1) effect of lower vs. higher Q; 2) effect of a fixed halo component; and 3) effect of a massive perturber.

To upload: Your plots, copies of your Python codes, and answers to the questions. The ICs and particle snapshots for $N = 10^5$ particles are large – please *don't* upload these!

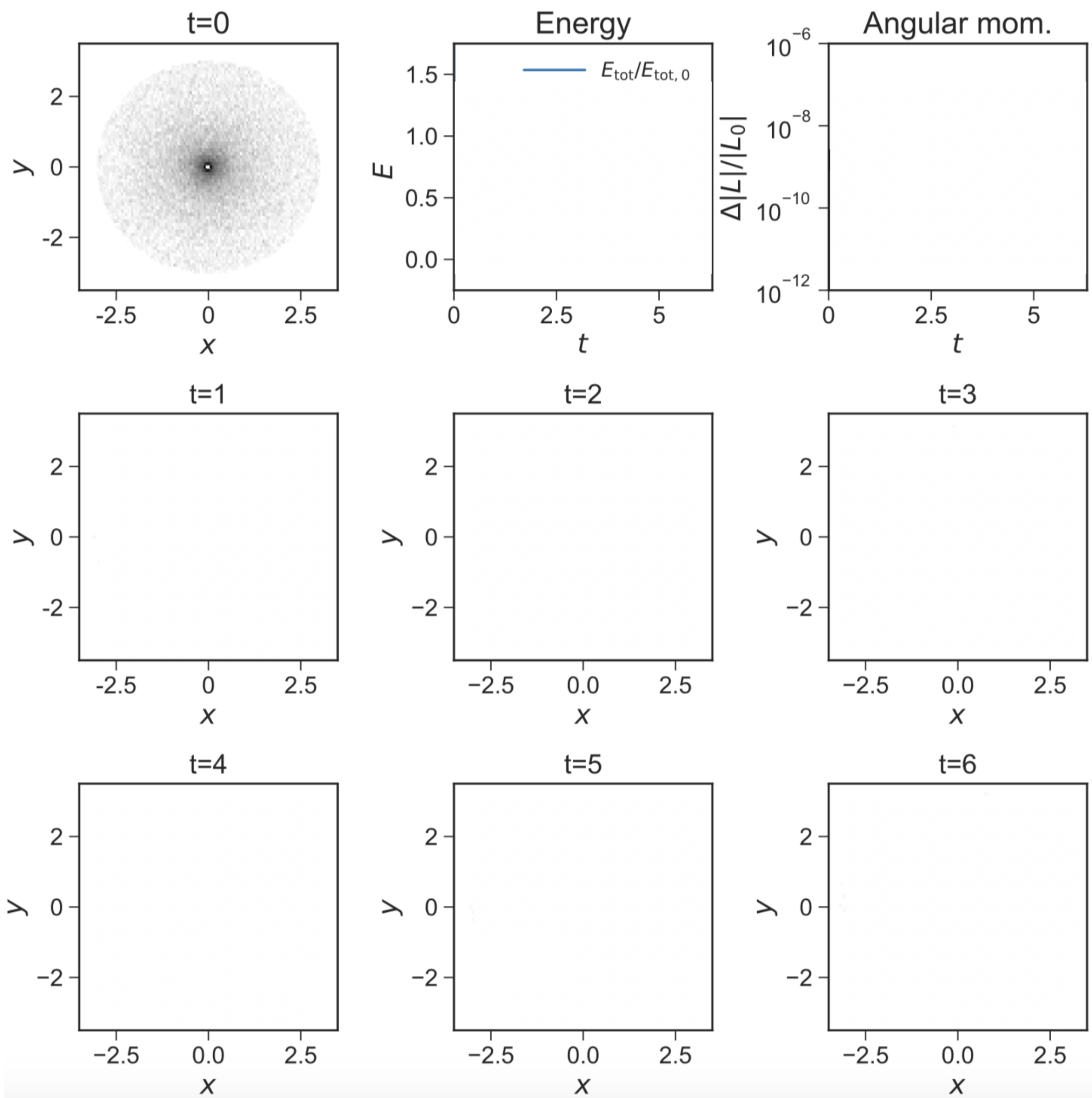

Fig. 2.— Example multi-panel plot to summarize the results of your disk simulations.